TIME-OPTIMAL PULSE OPERATION IN LINEAR SYSTEMS

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Optimal problems for linear system have been considered by many authors in connection with the problem of moments [1]. In [2] their solution is reduced to finding the minimax of certain known functions of special form, and in [3] to finding the maximum of a linear functional on a set which is itself determined from the maximum condition. Other modifications of the problem have also been investigated [4 and 5].

In the present paper, as an addendum to the results of [1] concerning the problem of time-optimal pulse operation, we propose to demonstrate the validity of the following statement: by virtue of the conditions set forth in the solution of [1], finding the minimax can be replaced by the maximum problem (Sections 2 and 3).

We shall also give an elementary proof of the statement of [4] concerning the number of controlling pulses (Section 3). A method for approximating solutions of linear differential equations by means of polynomials in order to simplify the computational side of the problem is described (Section 4).

1. Formulation of the problem. Let us consider the completely controlled system [6] described by Eq.

$$dy \,/\, dt = Ay + bu$$

where A is an $n \times n$ constant matrix; y and b are n-dimensional vectors; u is the scalar control.

By virtue of the complete controllability of the system, we can apply differentiation, elimination, and normalization of the control to form an equation in some linear combination x of phase coordinates

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = u \qquad (1.1)$$

We shall solve for Eq. (1.1) the problem of time-optimal motion from a given point (x_0, $x_0^{(1)}, \ldots, x_0^{(n-1)}$) to the origin on the set of all scalar controls with an integrable absolute value under the restriction

$$\int_0^\infty |u|\, dt \leqslant 1$$

Let us denote the matrix of the normal system of independent solutions of Eq. (1.1) for $u = 0$ by $V(t)$, and the instantaneous phase vector by $z(t)$,

$$V(t) = \left\| \begin{matrix} x_1(t) \ldots x_n(t) \\ \cdots\cdots\cdots \\ x_1^{(n-1)}(t) \ldots x_n^{(n-1)}(t) \end{matrix} \right\|, \qquad z(t) = \left\| \begin{matrix} x(t) \\ \cdots \\ x^{(n-1)}(t) \end{matrix} \right\|, \qquad z(0) = \left\| \begin{matrix} x(0) \\ \cdots \\ x^{(n-1)}(0) \end{matrix} \right\|$$

$$x_i^{(k)} \equiv \frac{d^k x_i}{dt^k}, \qquad x_i^{(k)}(0) = \delta_i^k$$

The group property of the solutions of differential equations implies, as we know, the identity $V^{-1}(t) = V(-t)$. From [2] we infer the following result.

The optimal control u^0 is a pulse control,

$$u^0 = \mu_1 \delta(t - t_1) + .. + \mu_r \delta(t - t_r]) \tag{1.2}$$

where $\delta(t - t_i)$ are delta functions.

The sum of absolute values of the controlling pulses μ_r is maximal, $|\mu_1| + ... + |\mu_r| = 1$.

The instants $t_1, ..., t_r$ of application of the pulses are determined by the solution of the problem

$$\min_{C_1,\ldots,C_n} \max_{0 \leqslant t \leqslant T} \left| \sum_{k=1}^{n} c_k x_n^{(k-1)}(-t) \right| = \left| \sum_{k=1}^{n} c_k^\circ x_n^{(k-1)}(t_j) \right| = 1 \tag{1.3}$$

under the condition

$$\sum_{k=1}^{n} c_k x_0^{(k-1)} = \sum_{k=1}^{n} c_k^\circ x_0^{(k-1)} = -1 \tag{1.4}$$

The optimal operating time T° is the smallest of all the T which satisfy not only (1.3), but also the "hit" conditions

$$-x_0^{(k)} = \sum_{i=1}^{r} x_n^{(k)}(-t_i)\mu_i \qquad (k = 0, \ldots, n-1) \tag{1.5}$$

We shall investigate conditions (1.2) to (1.5) with the aim of simplifying the actual synthesis of the optimal control.

We shall use the notation $x_n(-t) = \phi(t)$. The function $\phi(t)$ satisfies the differential Eq.

$$\varphi^{(n)} - a_1\varphi^{(n-1)} + \cdots + (-1)^n a_n \varphi = 0 \tag{1.6}$$

and the initial conditions

$$\varphi(0) = \cdots = \varphi^{(n-2)}(0) = 0, \qquad \varphi^{(n-1)}(0) = (-1)^{n-1}$$

Eqs. (1.3) and (1.5) then become

$$\min_{c_1,\ldots,c_n} \max_{0 \leqslant t \leqslant T} \left| \sum_{k=1}^{n} (-1)^{k-1} c_k \varphi^{(k-1)}(t) \right| = \left| \sum_{k=1}^{n} c_k^\circ (-1)^{k-1} \varphi^{(k-1)}(t_j) \right| = 1 \tag{1.7}$$

and

$$-x_0^{(k)} = \sum_{i=1}^{r} (-1)^k \varphi^{(k)}(t_i)\mu_i \tag{1.8}$$

respectively.

We note that the function

$$F(c, t) \equiv \sum_{k=1}^{n} (-1)^{k-1} c_k \varphi^{(k-1)}(t)$$

is the general solution of Eq. (1.6), since by virtue of the initial conditions for $\phi(t)$ at $t = 0$ the Wronskian

$$\begin{vmatrix} \varphi(0) \ldots \varphi^{(n-1)}(0) \\ \ldots\ldots\ldots\ldots \\ \varphi^{(n-1)}(0) \ldots \varphi^{(2n-3)}(0) \end{vmatrix} = \begin{vmatrix} 0 \quad \ldots (-1)^{n-1} \\ \ldots\ldots\ldots\ldots \\ (-1)^{n-1} \ldots \varphi^{(2n-3)}(0) \end{vmatrix} = (-1)^{n(n-1)} \tag{1.9}$$

is different from zero.

2. Ancillary propositions. 1°. We introduce the conditions

$$\operatorname{sign} F(c, t_j) = \operatorname{sign} \mu_j \tag{2.1}$$

for each $\mu_j \neq 0$, $t_1, \ldots, t_n$ and $t_1, \ldots, t_r$ which is a solution of problem (1.3). (These relations also appear in [4]).

L e m m a 2.1. Let Eqs. (1.7) and (1.8) be fulfilled. Then fulfillment of any two of the three conditions (1.2), (1.4), and (2.1) implies fulfillment of the third.

P r o o f. Multiplying each Eq. of (1.8) by c_{k+1} and summing over k, we obtain

$$-\sum_{k=0}^{n-1} c_{k+1}x_0^{(k)} = \sum_{i=1}^{r} \mu_i \sum_{k=0}^{n-1} c_{k+1}\varphi^{(k)}(t_i) = \sum_{i=1}^{r} \mu_i \operatorname{sign} F(c, t_i)$$

It is clear that for c_k and t_i which constitute a solution of problem (1.7) we have the relation $F(c, t_i) = \operatorname{sign} F(c, t_i)$. Thus,

$$-\sum_{k=0}^{n-1} c_{k+1}x_0^{(k)} = \sum_{i=1}^{r} \mu_i \operatorname{sign} F(c, t_i)$$

Let conditions (1.4) and (2.1) be fulfilled. Then

$$1 = -\sum_{k=0}^{n-1} c_{k+1}x_0^{(k)} = \sum_{i=1}^{r} \mu_i \operatorname{sign} F(c, t_i) = \sum_{i=1}^{r} \mu_i \operatorname{sign} \mu_i = \sum_{i=1}^{r} |\mu_i|$$

If conditions (1.4) and (2.1) are fulfilled, then

$$-\sum_{k=0}^{n-1} c_{k+1}x_0^{(k)} = \sum_{i=1}^{r} \mu_i \operatorname{sign} F(c, t_i) = \sum_{i=0}^{r} |\mu_i| = 1$$

Finally, let conditions (1.2) and (1.4) be fulfilled. We shall show that this implies Eqs. (2.1). Without limiting generality we can assume that among r numbers μ_i there are none which equal zero. We have

$$1 = -\sum_{k=0}^{n-1} c_{k+1}x_0^{(k)} = \sum_{i=1}^{r} |\mu_i| \delta_i \quad (\delta_i = \operatorname{sign} \mu_i \operatorname{sign} F(c, t_i))$$

Now let us assume that some (e.g. the first m) of the numbers δ are negative. Then

$$1 = \sum_{i=1}^{r} |\mu_i| \delta_i = -\sum_{i=1}^{m} |\mu_i| + \sum_{i=m+1}^{r} |\mu_i|$$

Eq. (1.2) then implies that $|\mu_1| + \ldots + |\mu_m| = 0$, which is impossible. Hence, $\delta_i = 1$.

2°. Let $c^\circ = (c_1^\circ, \ldots, c_n^\circ)$ be some fixed set of values c_i satisfying condition (1.4), and let $t_1^\circ < t_2^\circ < \ldots < t_r^\circ = T$ be those values of t_i for which

$$\max |F(c^\circ, t)| = |F(c^\circ, t_j^\circ)| = 1 \qquad (0 \leqslant t \leqslant T) \tag{2.2}$$

Let us consider the small ρ-neighborhood of the point c° defined by the conditions

$$|c_i - c_i^\circ| \leqslant \varepsilon_i < \rho \qquad (i = 1, \ldots, n)$$

Let L_j be the set of all values ε_i belonging to the above ρ-neighborhood for which $\max|F(c, t)|$ is attained for $0 \leqslant t = t_j \leqslant T$ which passes, by continuity, to t_j°, where $c = c^\circ$. Being the minimum or maximum point of $F(c, t)$, the value t_j is one of the solutions of Eq. $\partial F/\partial t = 0$ under the condition $\partial^2 F/\partial t^2 \neq 0$. From the theory of implicit functions it follows that the function $t_j = t_j(c)$ is continuous and has partial derivatives with respect to c_i if $\varepsilon \in L_j$.

To within ε_i^2 we have

$$F_j \equiv |F(c, t_j(c))| = \left| F(c^\circ, t_j^\circ) + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial c_i} + \frac{\partial F}{\partial t_j} \frac{\partial t_j}{\partial c_i} \right)\bigg|_{c=c^\circ} \varepsilon_i \right| =$$

$$= \left(\operatorname{sign} F(c^\circ, t_j^\circ) + \sum_{i=1}^{n} \frac{\partial F}{\partial c_i}\bigg|_{c=c^\circ} \varepsilon_i \right) \operatorname{sign} F(c^\circ, t_j^\circ) =$$

$$= 1 + \sum_{i=1}^{n} (-1)^{i-1}\varphi^{(i-1)}(t_j^\circ)\,\varepsilon_i\,\mathrm{sign}\,F(c^\circ, t_j^\circ) \equiv 1 + \Phi_j.$$

It is necessary that $t_r^\circ = T$, since otherwise a "hit" at the origin, which is a singular point, would be impossible.

Condition (2.2) implies that if $t^\circ \neq 0,\ T$, then $\partial F/\partial t = 0$. If either $t^0 = 0$ or $t^\circ = T$, then the instants are fixed and the derivative $\partial F/\partial t$ does not appear in the expressions for F_0 and F_r. Since by the definition of the set L_j we have

$$F_j = \max_{0\leqslant t\leqslant T} |F(c, t)| \quad \text{for} \quad \varepsilon \in L_j$$

it follows that

$$1 = \min_{c_1\ldots c_n} \max_{0\leqslant t\leqslant T} |F(c, t)| = \min_{\varepsilon_1,\ldots,\varepsilon_n} (1 + \Phi)$$

Here

$$\varepsilon_1 x_0 + \ldots + \varepsilon_n x_0^{(n-1)} = 0, \quad \Phi = \Phi_j \quad \text{for} \quad \varepsilon \in L_j\ (j = 1,\ldots, r)$$

The fact that the sets L_j are defined by the intersection of the manifolds $\Phi_j = 0$ linear in ε_i implies that the L_j form connected domains, each of which (provided it is nonempty) touches the origin ($\varepsilon = 0$). The totality of the domains L_j fills the entire ρ-neighborhood, so that the function $\max|F(c, t)|$ is defined everywhere in the ρ-neighborhood. Since no function can have two different values at the same point which are also maximum values, the function $\max|F(c, t)|$ is also single-valued. (This means that the domains L_j which do not coincide completely do not intersect in pairs). Finally, the continuity of the functions $t_j(c)$ implies the continuity in the ρ-neighborhood of the functions $\max|F(c, t)|$. The above facts imply the following Lemma.

L e m m a 2.2. Problem (2.3) breaks down into two independent problems, i.e.

a) the quantities c_i° and t_j° are determined by the conditions

$$\max_{0\leqslant t\leqslant T} \left| \sum_{i=1}^{n} (-1)^{i-1} c_i \varphi^{(i-1)}(t) \right| = \left| \sum_{i=1}^{n} (-1)^{i-1} c_i \varphi^{(i-1)}(t_j) \right| = 1 \tag{2.3}$$

$$c_1 x_0 + \cdots + c_n x_0^{(n-1)} = -1 \tag{2.4}$$

b) The resulting t_j° must satisfy the conditions

$$\min \Phi(\varepsilon) = 0 \qquad (|\varepsilon_i| \leqslant \rho) \tag{2.5}$$

$$\varepsilon_1 x_0 + \ldots + \varepsilon_n x_0^{(n-1)} = 0 \tag{2.6}$$

where $\Phi(\varepsilon)$ is a single-valued continuous function defined throughout the ρ-neighborhood by the conditions

$$\Phi(\varepsilon) = \sum_{i=1}^{n} (-1)^{i-1}\varphi^{(i-1)}(t_j^\circ)\,\varepsilon_i\,\mathrm{sign}\,F(c^\circ, t_j^\circ) \qquad (\varepsilon \in L_j)$$

Here for each p and $\varepsilon \in L_j$ we have the inequalities

$$\sum_{i=1}^{n} (-1)^{i-1}\varphi^{(i-1)}(t_j^\circ)\,\varepsilon_i\,\mathrm{sign}\,F(c^\circ, t_j^\circ) \geqslant \sum_{i=1}^{n} (-1)^{i-1}\varphi^{(i-1)}(t_p^\circ)\,\varepsilon_i\,\mathrm{sign}\,F(c^\circ, t_p^\circ)$$

3°. Let us establish the notation $a_{ji} = (-1)^{i-1} \phi^{i-1}(t_j^\circ)\ \mathrm{sign}\ F(c^\circ, t_j^\circ)$ (2.7) and consider the minimum of the function $\Phi(\varepsilon)$ defined by the conditions

$$\Phi(\varepsilon) = a_{j1}\varepsilon_1 + \cdots + a_{jn}\varepsilon_n \qquad (\varepsilon \in L_j) \qquad (j = 1, \ldots, r) \tag{2.7}$$

on linear manifold (2.6).

We begin by showing that the function $\Phi(\varepsilon)$ has a minimum if and only if the condition

$z > 0$ is fulfilled for arbitrary ε_i not simultaneously equal to zero, and for any z satisfying the condition $z \geqslant \Phi(\varepsilon)$.

Necessity follows immediately from the minimum condition: $\Phi(\varepsilon) > 0$ ($\varepsilon \neq 0$). Sufficiency can be proved indirectly. Let the inequality $z \geqslant \Phi(\varepsilon)$ imply that $z > 0$ for any ε_i and z, but let there exist a point ε^* for which $\Phi(\varepsilon^*) < 0$. Because z is arbitrary we can set $z = \Phi(\varepsilon^*) + \delta$, taking $\delta > 0$ sufficiently small. We then have $z < 0$ and $z \geqslant \Phi(\varepsilon)$, which is impossible.

Let us write

$$\eta_j = (a_{j1}\varepsilon_1 + \ldots + a_{jn}\varepsilon_n) - z \qquad (j = 1, \ldots, r) \tag{2.8}$$

What we have just proved implies that the function $\Phi(\varepsilon)$ has a minimum if and only if the inequality $z > 0$ follows from the system of conditions

$$\eta_1 \leqslant 0, \ldots, \eta_r \leqslant 0 \tag{2.9}$$

In fact, by virtue of (2.7) if $\varepsilon \in L_j$, then

$$\sum_{i=1}^{n} a_{ji}\varepsilon_i \geqslant \sum_{i=1}^{n} a_{ki}\varepsilon_i, \quad \text{or} \quad \eta_j \geqslant \eta_k \qquad (k \leqslant n)$$

If $z \geqslant \Phi(\varepsilon)$, then $\eta_j \leqslant 0$, so that $\eta_k \leqslant 0$. Conversely, the conditions $\eta_j \geqslant \eta_k$ and $\eta_j \leqslant 0$ which are valid for $\varepsilon \in L_j$ imply that $z \geqslant \Phi(\varepsilon)$.

Let h be the rank of the matrix $\|a_{ji}\|$. If $r \geqslant n = h$, it follows by (2.6) and (2.8) that the quantities $\eta_j + z$, and therefore η_j and z are related by exactly $r - n + 1$ independent linear relations. The inequality $z > 0$ evidently follows from conditions (2.9) if and only if η_j and z are related by at least one linear dependence with coefficients of the same sign. Let us consider three cases.

a) Let $r = n = h$. In this case η_j and z are related by just one linear dependence, i.e. by Eq.

$$\begin{vmatrix} a_{11} \ldots a_{1n} & \eta_1 + z \\ \ldots\ldots\ldots & \ldots \\ a_{n1} \ldots a_{nn} & \eta_n + z \\ x_0 \ldots x_0^{(n-1)} & 0 \end{vmatrix} = 0 \tag{2.10}$$

or

$$M_1\eta_1 + \ldots + M_n\eta_n + (M_1 + \ldots + M_n)\, z = 0$$

Here M_j is the algebraic complement of the j-th element of determinant (2.10). The coefficients of this linear bundle are of the same sign only if

$$M_i M_j \geqslant 0 \qquad (i, j = 1, \ldots, n) \tag{2.11}$$

Thus, conditions (2.11) are necessary and sufficient for the function $\Phi(\varepsilon)$ to have a minimum. Clearly, this minimum attained at the point $\varepsilon = 0$ is isolated only if all of relations (2.11) are strict inequalities. In fact, if one of the minors, e.g. M_1, is equal to zero, then to make z vanish we need merely set $\eta_2 = \ldots = \eta_n = 0$. These equations are clearly satisfied not only by zero values of ε_i.

In general the minimum of the function $\Phi(\varepsilon)$ attained at the point $\varepsilon = 0$ is isolated only if these does not exist a single linear relation with coefficients of the same sign relating fewer than n quantities $\eta_j + z$.

b) Let $r > n = h$. Evidently in this case there necessarily exist n quantities $\eta_{i1}, \ldots, \eta_{in}$ which together with (2.6) form a linear combination with coefficients of the same sign. This means that the function $\Phi(\varepsilon)$ has a minimum if and only if among the r quantities η_j there are n whose nonpositiveness implies the inequality $z > 0$. Solving any n Eqs. of (2.8), e.g. the first n Eqs. for $\eta_j + z$ and setting the result into the remaining Eqs. of (2.8) and (2.6), we obtain

$$\eta_{n+1} + z = \alpha_{11}(\eta_1 + z) + \ldots + \alpha_{1n}(\eta_n + z) \tag{2.12}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\eta_r + z = \alpha_{r-n,1}(\eta_1 + z) + \ldots + \alpha_{r-n,n}(\eta_n + z)$$
$$0 = \alpha_1(\eta_1 + z) + \ldots + \alpha_n(\eta_n + z)$$

Let us consider all the possible linear relations

$$A_1(\eta_1 + z) + \ldots + A_r(\eta_r + z) = 0 \tag{2.13}$$

fulfilled by virtue of system (2.12).

Substituting into (2.13) our expressions for $\eta_{n+1} + z, \ldots, \eta_r + z$ and equating to zero the coefficients of $\eta_1 + z, \ldots, \eta_n + z$, we find that

$$-A_\nu = A_{n+1}\alpha_{1\nu} + \cdots + A_r\alpha_{r-n,\nu} + \lambda\alpha_\nu \qquad (\nu = 1, \ldots, n) \tag{2.14}$$

If the function $\Phi^{(s)}$ has a minimum, then there exists a relation (2.13) in which all the A_k which constitute the solution of system (2.14) are of the same sign. For example, let $A_k = A_k^\circ > 0$. Sepcifying the quantities $A^\circ_{n+2}, \ldots, A^\circ_r$ in system (2.14) and reducing A_{n+1} beginning with A°_{n+1}, we find that at least one of the coefficients $A_1, \ldots, A_n, A_{n+1}$, e.g. A_{n+1}, vanishes before the others do. This means that there exist $A_2 > 0, \ldots, A_r > 0$

$$A_2(\eta_2 + z) + \ldots + A_r(\eta_r + z) = 0$$

where the sign of η_1 does not affect the sign of z.

Omitting the first Eq. in (2.8) and repeating the process $r - n$ times, we see the validity of the above statement.

c) Let $r < n$. Relations of the (2.13) type then yield the system of Eqs.

$$\begin{gathered} a_{11}A_1 + \cdots + a_{r_1}A_r + \lambda x_0 = 0 \\ \cdots\cdots\cdots\cdots \\ a_{1,n}A_1 + \cdots + a_{r,n}A_r + \lambda x_0^{(n-1)} = 0 \end{gathered} \tag{2.15}$$

The existence of nonzero solutions of this system requires that the rank h of the matrix $\|a_{ji}\|$ be equal to the rank of the matrix

$$\left\| \begin{matrix} a_{11} \ldots a_{r,1} & x_0 \\ \cdots\cdots & \\ a_{1n} \cdots a_{rn} & x_0^{(n-1)} \end{matrix} \right\| \tag{2.16}$$

and also that $h \leqslant r$. First let $h = r$ and det $a_{\lambda\mu} \neq 0$ $(\lambda, \mu = r)$.

Solving the first r Eqs. of (2.15) for A_k and requiring that they all be of the same sign, we obtain the conditions

$$(-1)^{i+j} M_i{}^* M_j{}^* \geqslant 0 \qquad (i, j = 1, \ldots, r) \tag{2.17}$$

which are similar to conditions (2.11). Let $M_j{}^*$ be the minor of the matrix (2.16) which we obtain by crossing out the j-th column from the latter. If $h < r$ and det $a_{\lambda\mu} \neq 0$ $(\lambda, \mu = 1, \ldots, h)$ we obtain a similar result in which the role of matrix (2.16) is played by the matrix

$$\left\| \begin{matrix} a_{11} \ldots a_{h1} & x_0 \\ \cdots\cdots & \\ a_{1h} \cdots a_{hh} & x_0^{(h-1)} \end{matrix} \right\|$$

In both cases ($h = r$, $h < r$) the function $\Phi(s)$ has a minimum on the $(n - h)$-dimensional linear manifold containing the point $s = 0$. Thus, the following Lemma is valid.

Lemma 2.3. Let the rank of the matrix

$$\|a_{ij}\| = \left\| \begin{matrix} a_{11} \ldots a_{r_1} \\ \cdots\cdots \\ a_{1n} \ldots a_{rn} \end{matrix} \right\|$$

be h. This matrix then contains h rows (if $h \leqslant n$) or h columns (if $r \geqslant n$) (e.g. the first h rows or columns) from which we can construct the matrix.

$$\left\| \begin{matrix} 1 \dots 1 & 0 \\ a_{11} \dots a_{h,1} & x_0 \\ \dots\dots\dots & \dots \\ a_{1,h} \dots a_{h,h} & x_0^{(h-1)} \end{matrix} \right\| \tag{2.18}$$

which has the following property (*): the function $\Phi(\varepsilon)$ has a minimum if and only if the pairwise products of the algebraic complements of the elements of its first row are nonnegative. This minimum attained at the point $\varepsilon = 0$ is isolated (**) only in the case $h = n$.

3. Reduction of the problem.

Let the rank of the matrix

$$\| a_{ij} \| = \| (-1)^{i-1} \varphi^{(i-1)} (t_j^\circ) \operatorname{sign} F(c^\circ, t_j^\circ) \|$$

be n. We must find an expression for the minor M_j^* of the matrix

$$\left\| \begin{matrix} \varphi(t_1^\circ) \operatorname{sign} F(c^\circ, t_1^\circ) \dots & \varphi(t_n^\circ) \operatorname{sign} F(c^\circ, t_n^\circ) & x_0 \\ (-1)^{n-1} \varphi^{(n-1)} (t_1^\circ) \operatorname{sign} F(c^\circ, t_1^\circ) \dots & (-1)^{n-1} \varphi^{(n-1)} (t_n^\circ) \operatorname{sign} F(c^\circ, t_n^\circ) & x_0^{(n-1)} \end{matrix} \right\|$$

obtained by crossing out its j-th column. Substituting in $x_0^{(k)}$ from (1.8), we obtain

$$M_1^* = (-1)^n D \mu_1 \operatorname{sign} F(c^\circ, t_2^\circ) \dots \operatorname{sign} F(c^\circ, t_n^\circ)$$

Here D is the determinant of the matrix $\|a_{ij}\|$. Similarly, (3.1)

$$M_j^* = (-1)^{n-j+1} D \operatorname{sign} F(c^\circ, t_1^\circ) \dots \operatorname{sign} F(c^\circ, t_{j-1}^\circ) \mu_j \operatorname{sign} F(c^\circ, t_{j+1}^\circ) \dots \operatorname{sign} F(c^\circ, t_n^\circ)$$

Theorem 3.1. On the number of pulses (***). The number r of pulses μ_i effecting time-optimal operation does not exceed the dimensionality of the problem

$$r \leqslant n \tag{3.2}$$

Proof. Let us show that if $r > n$, then either the time T is not optimal or time-optimal operation is also realizable with a number of pulses $r \leqslant n$. In fact, let $r > n$. By Lemma 2.3 the conditions whereby $\Phi(\varepsilon)$ has a minimum are that

$$M_i M_j = (-1)^{i+j} M_i^* M_j^* \geqslant 0 \tag{3.3}$$

for each pair of minors M^* of matrix (2.22).

First let $h = n$. Let us define the new values μ_i' of the pulses by Formulas

$$-x_0^{(k)} = \sum_{i=1}^{n} (-1)^k \varphi^{(k)} (t_i) \mu_i', \qquad \mu_{n+1}' = \dots = \mu_r' = 0 \tag{3.4}$$

Making use of (2.1), (3.1), and (3.3), we obtain

$$M_i M_j = (-1)^{2n-2} D^2 (\mu_i' \operatorname{sign} \mu_i)(\mu_j' \operatorname{sign} \mu_j) \geqslant 0 \tag{3.5}$$

Hence,

$$(\mu_i' \operatorname{sign} \mu_i)(\mu_j' \operatorname{sign} \mu_j) \geqslant 0.$$

On the other hand, conditions (2.3) and (2.1) imply fulfillment of Eqs.

$$\begin{gathered} c_1 \varphi(t_1) + \dots + (-1)^{n-1} c_n \varphi^{(n-1)} (t_1) = \operatorname{sign} \mu_1 \\ \dots\dots\dots\dots\dots\dots \\ c_1 \varphi(t_n) + \dots + (-1)^{n-1} c_n \varphi^{(n-1)} (t_n) = \operatorname{sign} \mu_n \\ c_1 x_0 + \dots + c_n x_0^{(n-1)} = -1 \end{gathered}$$

From this we obtain

$$\left| \begin{matrix} \varphi(t_1) \dots (-1)^{n-1} \varphi^{(n-1)} (t_1) & \operatorname{sign} \mu_1 \\ \dots\dots\dots\dots & \dots \\ \varphi(t_n) \dots (-1)^{n-1} \varphi^{(n-1)} (t_n) & \operatorname{sign} \mu_n \\ x_0 \quad\quad x_0^{(n-1)} & -1 \end{matrix} \right| = 0$$

*) Here and below we assume that matrices of the (2.18) type do not necessarily contain the first columns (rows) of the matrix $\|a_{ij}\|$; we consider them to have been renumbered.

(*Footnotes continued on next page*)

By virtue of (3.4) this is equivalent to Eq.

$$\begin{vmatrix} \varphi(t_1) \ldots (-1)^{(n-1)}\varphi^{(n-1)}(t_1) \\ \cdots\cdots\cdots \\ \varphi(t_n) \ldots (-1)^{n-1}\varphi^{(n-1)}(t_n) \end{vmatrix} (1 - \mu_1' \operatorname{sign} - \cdots - \mu_n' \operatorname{sign}_n) = 0$$

Hence,

$$\mu_1' \operatorname{sign} \mu_1 + \ldots + \mu_n' \operatorname{sign} \mu_n = 1 \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain

$$\mu_j' \operatorname{sign} \mu_j > 0 \quad (j = 1,\ldots, n), \qquad \text{or} \qquad \operatorname{sign} \mu_j' = \operatorname{sign} \mu_j$$

Thus, if we know a time-optimal operating mode with $r > n$ pulses occurring at the instants $t_1,\ldots, t_r$, then from these instants we can choose n (they are denoted by $t_1,\ldots, t_n$ in the proof) and set μ_j equal to zero at the remaining instants, so that all the time optimality conditions, i.e. (1.5), (2.1), and (2.3) to (2.6), are fulfilled.

If it turns out here that $t_n < T$, then the initially determined T is not optimal. For $h < n$ we must carry out the proof using linear dependences among the elements of the matrix $\|a_{ij}\|$ as is done below in the proof of Theorem 3.3. The proof remains unchanged in other respects. Theorem has been proved.

The above analysis of conditions (1.2) to (1.5) enables us to formulate the following result.

Theorem 3.2. The optimal control in the time-optimal operation problem for Eq. (1.1) is a pulse control. The sum of absolute values of the controlling pulses μ_j is maximal,

$$|\mu_1| + \ldots + |\mu_r| = 1$$

The instants $t_1,\ldots, t_r$ when the pulses are applied are-determined by the solution of Problem (2.3).

The optimal operating time T° is the smallest of all the T which satisfy (2.3), hit conditions (1.8) where $k = 0,\ldots, n-1$, and conditions (2.1) for the signs of the pulses for each $\mu_j \neq 0$. The number r of nonzero pulses satisfies the inequalities

$$h \leqslant r \leqslant n$$

where h is the rank of the expanded matrix of system (1.8).

Proof. According to Lemmas 2.1 and 2.2, conditions (2.3), (2.1), and (1.2) replace condition (1.4). The condition $r \leqslant n$ is proved in Theorem 3.1. The inequality $h \leqslant r$ is the condition of solvability of system (1.8). Taking into account Lemma 2.2, we must show that conditions (2.5) and (2.6) are satisfied by virtue of the conditions of our theorem. Let us show this.

We consider matrix (2.18) of Lemma (2.3) which we shall have occasion to use below.

Let its rank r be n. With allowance for conditions (2.1), Formulas (3.1) become

$$M_j^* = (-1)^{n-j+1} D \operatorname{sign} \mu_1 \ldots \operatorname{sign} \mu_{j-1} \mu_j \operatorname{sign} \mu_{j+1} \ldots \operatorname{sign} \mu_n$$

Then

$$M_i M_j = (-1)^{i+j} M_i^* M_j^* = D^2 \mu_i \operatorname{sign} \mu_i \mu_j \operatorname{sign} \mu_j = D^2 |\mu_i| \, |\mu_j| \geqslant 0$$

According to Lemma 2.3 this means that the function $\Phi(\varepsilon)$ has a minimum (condition (2.5)) on manifold (2.6).

Now let $r = h < n$. By agreement, the matrix

$$\begin{Vmatrix} \varphi(t_1) & \ldots & \varphi(t_h) & x_0 \\ \cdots & \cdots & \cdots & \cdots \\ (-1)^{n-1}\varphi^{(n-1)}(t_1) & \ldots & (1)^{n-1}\varphi^{(n-1)}(t_h) & x_0^{(h-1)} \end{Vmatrix}$$

contain h linearly independent rows (e.g. the first h rows). There exist numbers $\alpha_1^l,\ldots, \alpha_n^l$ such that the relations

(Footnotes continued from previous page)

**) Lemmas 2.2 and 2.3 are related to the results of K. Carathéodory and N.G. Chebotarev[7].

***) See also [3].

$$x_0^{(l)} = \alpha_1{}^l x_0 + \cdots + \alpha_h{}^l x^{(h-1)} \qquad (l = h+1, \ldots, n)$$

$$(-1)^l \varphi^{(l)}(t_i) = \alpha_1{}^l \varphi(t_i) + \cdots + \alpha_h{}^l (-1)^{h-1} \varphi^{(h-1)}(t_i) \qquad (i = 1, \ldots, h)$$

are valid.

We have

$$\Phi_i = \sum_{j=1}^{n} (-1)^{j-1} \varepsilon_j \varphi^{(j-1)}(t_i) = \sum_{j=1}^{h} (-1)^{i-1} \varepsilon_j \varphi^{(j-1)}(t_i) +$$

$$+ \sum_{l=h+1}^{n} (-1)^{l-1} \varepsilon_l \varphi^{(l-1)}(t_i) = \sum_{j=1}^{h} (-1)^{j-1} E_j \varphi^{(j-1)}(t_i)$$

where

$$E_j = \varepsilon_j + \sum_{l=h+1}^{n} \alpha_j{}^{l-1} \varepsilon_l \; (j = 1, \ldots, h)$$

Similarly,

$$\varepsilon_1 x_0 + \cdots + \varepsilon_n x_0^{(n-1)} = \sum_{j=1}^{h} E_j x_0^{(j-1)}$$

Now we need merely cite the first part of the proof of this theorem and Lemma 2.3.

4. Approximation of the function $F(c, t)$ by polynomials. From Theorem 3.2 we see that the principal difficulty of the initial problem lies in choosing the parameters c_i of the function

$$F(c, t) = \sum_{i=1}^{n} (-1)^{i-1} c_i \varphi^{(i-1)}(t)$$

in such a way that the curve of this function on the segment $[0, T]$ lies inside the strip $|F(c, t)| \leqslant 1$ and touches its boundaries the required number of times $r \leqslant n$.

The fact that we can here ignore the behavior of the function $F(c, t)$ outside the above strip enables us to approximate it by the method described below. The basic purpose of this approximation is to provide a means of effective computation of the instants t_j of application of the pulses (i.e. of the points for which $F(c, t_j) = 1$).

Let us write

$$F(c, t) = f(t) = f, \quad \frac{\partial^k F(c, t)}{\partial t^k} = f^{(k)}(t) = f^{(k)}, \quad f^{(k)}(0) = f_0^{(k)}$$

where f is the solution of the differential Eq.

$$f^{(n)} = a_1 f^{(n-1)} + \cdots + (-1)^{n-1} a_n f \tag{4.1}$$

Integrating Eq. (4.1) n times from 0 to t and transforming, we obtain

$$f = f_0 + \sum_{s=1}^{n-1} \Big(f_0^{(s)} - \sum_{k+j=s} (-1)^{k-1} a_k f_0^{(j)} \Big) \frac{t^s}{s!} + \sum_{k=1}^{n} (-1)^{k-1} a_k \int_k f\, dt \tag{4.2}$$

where the subscript k at the integral sign denotes integration over t. If $1 \leqslant f \leqslant -1$ on the segment $[0, T]$ we obtain

$$(-1)^{k-1} a_k \int_k f\, dt \leqslant (-1)^{k-1} a_k \int_k dt = (-1)^{k-1} a_k \frac{t^k}{k!} \quad \text{for} \quad (-1)^{k-1} a_k \geqslant 0$$

$$(-1)^{k-1} a_k \int_k f\, dt \geqslant (-1)^{k-1} a_k \int_k dt = (-1)^{k-1} a_k \frac{t^k}{k!} \quad \text{for} \quad (-1)^{k-1} a_k < 0$$

Hence,

$$-|a_k|\frac{t^k}{k!}\leqslant(-1)^{k-1}a_k\int\limits_k f\,dt\leqslant|a_k|\frac{t^k}{k!}$$

For $0\leqslant t\leqslant T$ this yields

$$P_{1,n}^-\equiv\Psi_{1,n}-\sum_{k=1}^{n}|a_k|\frac{t^k}{k!}\leqslant f\leqslant\Psi_{1,n}\sum_{k=1}^{n}|a_k|\frac{t^k}{k!}\equiv P_{1,n}^+ \tag{4.3}$$

$$\Psi_{1,n}=f_0+\sum_{s=1}^{n}\Big(f_0^{(s)}-\sum_{k+j=s}(-1)^{k-1}a_kf_0^{(j)}\Big)\frac{t^s}{s!}$$

If we take the polynomial $\Psi_{1,n}=\frac{1}{2}(P_{1,n}^{+}+P_{1,n}^{-})$ as our first approximation for f, the error is given by

$$\Delta_{1,n}=|f-\Psi_{1,n}|=\sum_{k=1}^{n}|a_k|\frac{t^k}{k!} \tag{4.4}$$

Making use of estimate (4.3) of the lower and upper bounds, we can construct the next approximation and any number of subsequent ones.

Let $P_{m,n}^{+}\leqslant f\leqslant P_{m,n}^{-}$, where $P_{m,n}$ are polynomials of degree mn; n is the number of the approximation.

Let us define the functions

$$\xi_1^+=\tfrac{1}{2}[1-(-1)^{k-1}\operatorname{sign}a_k]\,P_{m,n}^{+}+\tfrac{1}{2}[1+(-1)^{k-1}\operatorname{sign}a_k]\,P_{m,n}^{-}$$

$$\xi_2=\tfrac{1}{2}[1+(-1)^{k-1}\operatorname{sign}a_k]\,P_{m,n}^{+}+\tfrac{1}{2}[1-(-1)^{k-1}\operatorname{sign}a_k]\,P_{m,n}^{-}$$

Then clearly,

$$(-1)^{k-1}a_k\int\limits_k\xi_1dt\leqslant(-1)^{k-1}a_k\int\limits_k f\,dt\leqslant(-1)^{k-1}a_k\int\limits_k\xi_2dt \tag{4.5}$$

Simplifying, we obtain

$$P_{m+1,n}^{-}\equiv\Psi_{m+1,n}-\frac{1}{2}\sum_{k=1}^{n}|a_k|\int\limits_k(P_{m,n}^{+}-P_{m,n}^{-})\,dt\leqslant f\leqslant\Psi_{m+1,n}+$$

$$+\frac{1}{2}\sum_{k=1}^{n}|a_k|\int\limits_k(P_{m,n}^{+}-P_{m,n}^{-})\,dt\equiv P_{m+1,n}^{+}$$

where

$$\Psi_{m+1,n}=f_0+\sum_{s=1}^{n-1}\Big(f_0^{(s)}-\sum_{k+j=s}(-1)^{k-1}a_kf_0^{(j)}\Big)\frac{t^s}{s!}+$$

$$+\frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}a_k\int\limits_k(P_{m,n}^{+}-P_{m,n}^{-})\,dt$$

If we assume that $f\approx\Psi_{m,n}$, this yields the following recurrent formula for the approximation error:

$$\Delta_{m+1,n}\equiv\frac{P_{m+1,n}^{+}-P_{m+1,n}^{-}}{2}=\sum_{k=1}^{n}|a_k|\int\limits_k\Delta_{m,n}\,dt$$

Let us show that as m increases for any fixed n this error tends to zero uniformly with t on the segment $[0, T]$ i.e. that

$$\Delta_{1,n}=\sum_{k=1}^{n}|a_k|\frac{t^k}{k!}\leqslant at\sum_{k=1}^{n}\frac{t^{k-1}}{k!}\leqslant ate^t\leqslant ae^Tt,\qquad a=\max\{|a_k|\}$$

In general, if

$$\Delta_{m,n} \leqslant \frac{(ae^T)^m}{m!} |t^m \tag{4.6}$$

then

$$\Delta_{m+1,n} = \sum_{k=1}^{n} |a_k| \int_k \Delta_{m,n} dt \leqslant \frac{(ae^T)^m}{m!} \sum_{k=1}^{n} |a_k| \int_k t^m \, dt =$$

$$= \frac{(ae^T)^m}{m!} a \sum_{k=1}^{n} \frac{t^{k+m}}{(m+1)\ldots(m+k)} = \frac{(ae^T)^m}{(m+1)!} at^{m+1} \sum_{k=1}^{n} \frac{t^{k-1}}{(m+2)\ldots(m+k)} \leqslant$$

$$\leqslant \frac{(ae^T)^{m+1}}{(m+1)!} t^{m+1}$$

Thus, for $0 \leqslant t \leqslant T < \infty$ we have

$$\Delta_{m,n}(t) \leqslant \frac{(aTe^T)^m}{m!}$$

This means that $\Delta_{m,n} \to 0$ as $m \to \infty$. The coarseness of the above estimates shows that, in fact, the approximations converge much more rapidly for $T < \infty$. If a_k and T are small, then it is enough to use the first approximation.

Let us find the function which approximates the $\phi(t)$ appearing in $F(c, t) = f(t)$ when f is approximated by the polynomial $\Psi_{1,n}$.

The formulas for converting from $f_0^{(k)}$ to c_i are of the form

$$c_i = (-1)^{i-1} \sum_{j=0}^{n-i} (-1)^j a_j f_0^{(n-j-i)} \cdot$$

Hence,

$$c_{n-s} = (-1)^{n-s} \sum_{k+j=s} (-1)^{k-1} a_k f_0^{(j)}$$

so that

$$f \approx \sum_{s=0}^{n-1} (-1)^{n-s} c_{n-s} \frac{t^s}{s!}, \qquad \varphi \approx \frac{t^{n-1}}{(n-1)!}$$

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